

Poincaré-Bendixson Theorem

(and related results)

In this section we will be concerned with **two-dimensional** dynamical systems described by a (nonlinear) system of first-order ODEs,

$$\dot{x}_1 = F_1(x_1, x_2), \quad \dot{x}_2 = F_2(x_1, x_2), \quad (1.1)$$

where $x_j \equiv x_j(t)$ for $j = 1, 2$ ($t \geq t_0$). We can express this autonomous system of differential equations in a more compact form by using vector notation,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \quad (1.2)$$

with

$$\mathbf{x} \equiv \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{F}(\mathbf{x}) \equiv \begin{bmatrix} F_1(x_1, x_2) \\ F_2(x_1, x_2) \end{bmatrix}.$$

We have seen (e.g., by using PPLANE) that each initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, for some $\mathbf{x}_0 \in \mathbb{R}^2$, produces a solution $\mathbf{x}(t)$ of (1.2), which geometrically represents a parametric curve

$$t \rightarrow \mathbf{x}(t; \mathbf{x}_0), \quad (\text{with } t \geq t_0)$$

in the phase plane. Here, we have included the dependence of $\mathbf{x}(t)$ on the initial condition since different choices for \mathbf{x}_0 will lead to different curves in the phase plane. Such curves will be referred to as the **trajectories** or **paths** of our dynamical system generated by the planar system (1.2); whenever possible we shall use the notation \mathcal{P} for such paths.

1.1 Stability

In what follows it will be assumed that $F_1(0, 0) = F_2(0, 0) = 0$, i.e. the origin $\mathbf{0} = (0, 0)$ represents an *equilibrium point* for the dynamical system generated by (1.2).

Definition 1: Let $D \subset \mathbb{R}^2$ be an open region that contains the origin. The function $V = V(\mathbf{x})$ is said to be **positive definite** in D if the following two conditions are satisfied:

1. $V(\mathbf{x}) \geq 0$ for all $\mathbf{x} = (x_1, x_2) \in D$;
2. $V(\mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

A **negative definite** function is one which satisfies the same requirements, except that the sense of the inequality in the first condition above is reversed. The next definitions clarify the various flavours of stability associated with the possible equilibrium points of

the system (1.2).

Definition 2: Let x_* be an equilibrium point of a (1.2), i.e. $\mathbf{F}(x_*) = \mathbf{0}$. This is said to be **stable** if:

$$(\forall) \varepsilon > 0 \quad (\exists) \delta = \delta(\varepsilon) > 0 \quad \text{such that} \quad \|\mathbf{x}(t; \mathbf{x}_0) - \mathbf{x}_*\| < \varepsilon, \quad (\forall) t \geq t_0,$$

provided that $\|\mathbf{x}_0 - \mathbf{x}_*\| < \delta$.

Definition 3: The equilibrium point \mathbf{x}_* is said to be **asymptotically stable** if it is stable and

$$\mathbf{x}(t; \mathbf{x}_0) \rightarrow \mathbf{x}_* \quad \text{as} \quad t \rightarrow \infty, \quad (\forall) \mathbf{x}_0 \quad \text{such that} \quad \|\mathbf{x}_0 - \mathbf{x}_*\| < \delta.$$

Definition 4: The equilibrium point \mathbf{x}_* is said to be **globally asymptotically stable** if it is stable and

$$\mathbf{x}(t; \mathbf{x}_0) \rightarrow \mathbf{x}_* \quad \text{as} \quad t \rightarrow \infty, \quad (\forall) \mathbf{x}_0 \in \mathbb{R}^2.$$

The next concept is of significant importance in the theory of stability. Before we can state a precise definition some preliminary terminology must be introduced. To this end, let us assume that $V(x_1, x_2)$ is a function that has continuous partial derivatives. If $\mathbf{x}(t) \equiv (x_1(t), x_2(t))$ is a solution of (1.2) – for some initial condition \mathbf{x}_0 , one can consider the total time derivative

$$\dot{V} \equiv \frac{d}{dt} V(x_1(t), x_2(t)),$$

which can be easily computed by using the chain rule and taking into account that $\dot{x}_j = F_j$ for $j = 1, 2$. This gives

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial V}{\partial x_2} \frac{dx_2}{dt} = \frac{\partial V}{\partial x_1} \dot{x}_1 + \frac{\partial V}{\partial x_2} \dot{x}_2 \\ &= \frac{\partial V}{\partial x_1} F_1 + \frac{\partial V}{\partial x_2} F_2 \\ &= (\nabla V) \cdot \mathbf{F}, \end{aligned}$$

where on the last line we have the scalar product between the gradient of the scalar field V and the right-hand side of (1.2).

Definition 5:

1. Let V be a function as explained above. We say that V is a **Lyapunov function** for (1.2) if $\dot{V}(x_1, x_2) \leq 0$ in some open region that contains the origin $\mathbf{0} \in \mathbb{R}^2$.
2. V is said to be a **strict Lyapunov function** if in addition to the above condition we also have

$$\dot{V}(x_1, x_2) = 0 \quad \iff \quad (x_1, x_2) = (0, 0).$$

Theorem 1 (Lyapunov): Consider (1.2) under the assumption that $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ (i.e., $\mathbf{x}_* = \mathbf{0}$ is an equilibrium point for the corresponding dynamical system).

1. The zero solution is *stable* if the system (1.2) has a Lyapunov function.
2. The zero solution is *asymptotically stable* if the system (1.2) has a strict Lyapunov function.

The proof of this important result is beyond the scope of this module, but the essential idea is that $t \rightarrow V(x_1(t), x_2(t))$ increases or decreases monotonically with t if \dot{V} is positive or negative. Because V is positive definite, $\mathbf{x}(t)$ decreases or increases according to whether V decreases or increases everywhere near $\mathbf{0}$.

Finding a Lyapunov function can be quite challenging for complicated problems. We look at a couple of examples that are representative for the type of problems that we will be concerned with in this module.

Example 1: Using Lyapunov's Theorem, discuss the stability of the origin for the system

$$\dot{x}_1 = x_1x_2^2 - x_1, \quad \dot{x}_2 = -2x_1^2x_2. \quad (1.3)$$

SOLUTION: We are going to look for a Lyapunov function of the form

$$V(x_1, x_2) = Ax_1^2 + Bx_2^2,$$

where $A, B > 0$ are to be found.

In this case it is clear that $F_1 \equiv x_1x_2^2 - x_1$ and $F_2 \equiv -2x_1^2x_2$, so

$$\dot{V}(x_1, x_2) = (2A - 4B)x_1^2x_2^2 - 2Ax_1^2.$$

By choosing $A = 2B$ and $B = 1$, we find that $V = 2x_1^2 + x_2^2$ and $\dot{V} = -4x_1^2 \leq 0$. Thus, we conclude that the origin $\mathbf{0} = (0, 0)$ is a stable equilibrium for (1.3).

Example 2: Using Lyapunov's Theorem discuss the stability of the origin for the system

$$\dot{x}_1 = x_1x_2^2 - x_1, \quad \dot{x}_2 = -x_2 - 2x_1^2x_2. \quad (1.4)$$

SOLUTION: Since this is very similar to the previous one, we can start directly with

$$V(x_1, x_2) = 2x_1^2 + x_2^2. \quad (1.5)$$

Routine calculations show that

$$\dot{V} = -2(2x_1^2 + x_2^2) \leq 0.$$

Since V is a strict Lyapunov function in this case – see (1.5), it can be concluded that the origin $\mathbf{0} = (0, 0)$ is an asymptotically stable equilibrium point for (1.4).

For linear systems of ODEs with *constant* coefficients the stability properties of equilibrium points can be inferred directly by examining the eigenvalues of the coefficient matrix, so the Lyapunov's Theorem is superfluous in that context. Nevertheless, it is still applicable as the next example shows.

Example 3: Show that the origin is an asymptotically stable equilibrium point for the system

$$\dot{x}_1 = ax_1 - cx_2, \quad \dot{x}_2 = cx_1 + ax_2, \quad (a < 0). \quad (1.6)$$

SOLUTION: Let V be the function

$$V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2).$$

In principle, we can use Theorem 1 to solve this problem, etc. However, here we adopt a slightly different approach that clarifies better the origin of the asymptotic stability mentioned above. Note that

$$\begin{aligned} \frac{dV}{dt} &= x_1 \frac{dx_1}{dt} + x_2 \frac{dx_2}{dt} \\ &= (ax_1^2 - cx_1x_2) + (cx_1x_2 + ax_2^2) \\ &= 2aV \leq 0. \end{aligned}$$

The first-order separable ODE $\dot{V} = 2aV$ can be solved in closed form to give

$$V(x_1(t), x_2(t)) = V(\mathbf{x}_0)e^{2at},$$

where we have assumed that the solution of (1.6) satisfies $\mathbf{x}(0) = \mathbf{x}_0$. Since $a < 0$, it is clear that $e^{2at} \rightarrow 0$ as $t \rightarrow \infty$, and therefore

$$V(x_1(t), x_2(t)) \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

Because V is positive definite, it also follows that $x_1(t), x_2(t) \rightarrow 0$ as $t \rightarrow \infty$, which is the same as the *global* asymptotic stability of the zero solution for (1.6).

Example 4: Show that the origin is an asymptotically stable equilibrium point for the system

$$\dot{x}_1 = -x_1 - 2x_2^2, \quad \dot{x}_2 = x_1x_2 - x_2^3. \quad (1.7)$$

SOLUTION: Let us consider the function

$$V(x_1, x_2) = \frac{1}{2}(x_1^2 + Bx_2^2),$$

where $B > 0$ will be determined as we go along. Note that for this choice of V ,

$$\dot{V} = -x_1^2 - Bx_2^4 + x_1x_2^2(B - 2).$$

Setting $B = 2$ it follows that $\dot{V} = -(x_1^2 + 2x_2^4) \leq 0$, so the origin is an asymptotically stable equilibrium point for (1.7).

OBSERVATIONS: Before we move on to the main part of this chapter, we pause to record below a number of observations that will turn out to be very useful in the next section. We are interested to know the direction in which the trajectories of (1.2) cross the boundary of a closed simple curve which represents the **level set** of a positive definite

function $V(x_1, x_2)$. For example, in the case of $V = x_1^2 + x_2^2$, such level sets correspond to a family of concentric *circles* centred at the origin,

$$\mathcal{C}_\alpha := \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = \alpha\},$$

while for $V = Ax_1^2 + Bx_2^2$ ($A, B > 0$), the level sets represent a family of concentric *ellipses* centred at the origin,

$$\mathcal{E}_\alpha := \{(x_1, x_2) \in \mathbb{R}^2 : Ax_1^2 + Bx_2^2 = \alpha\}.$$

We will assume that \mathcal{P} a generic path of (1.2) described by the parametric equations

$$x_1 = x_1(t), \quad x_2 = x_2(t), \quad (t \geq 0).$$

Let $M = M(x_1, x_2)$ be the intersection point between the trajectory \mathcal{P} and the level curve \mathcal{T} (e.g., $\mathcal{T} \equiv \mathcal{C}_\alpha$, etc), and recall that $\dot{V} \equiv (\nabla V) \cdot \mathbf{F}$.

- If $\dot{V}|_M > 0$ then \mathcal{P} points outward from \mathcal{T} .
- If $\dot{V}|_M < 0$ then \mathcal{P} points inward across \mathcal{T} .
- If $\dot{V}|_M = 0$ then \mathcal{P} is tangential to \mathcal{T} .

1.2 The existence of periodic trajectories

Theorem 2 (Poincaré-Bendixson): Consider the planar dynamical system (1.2) and let $\mathcal{R} \subset \mathbb{R}^2$ be a closed, bounded region which contains no singular points of (1.2)¹. Assuming that there exists a trajectory \mathcal{P} of the aforementioned dynamical system that remains in \mathcal{R} for all $t \geq 0$, the following (mutually exclusive) scenarios are possible:

1. \mathcal{P} is itself a closed trajectory; or
2. \mathcal{P} spirals toward a closed trajectory; or
3. \mathcal{P} terminates at an equilibrium point as $t \rightarrow \infty$.

This result can be used to prove the *existence* of **limit cycles** for two-dimensional autonomous systems of the form (1.2). Two things are required. First, one needs to ensure that the third possibility in Theorem 2 is ruled out right from the outset. This is most easily done by first solving the equation $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ to identify the possible equilibrium points, and then choosing \mathcal{R} so that it does not contain any of those points. The remaining two options in Theorem 2 tell us that there is at least one closed trajectory in \mathcal{R} , provided that the solutions starting in that region cannot “escape” from it. Such a region can be constructed relatively easily for the type of dynamical systems we will be

¹A point \mathbf{x}_0 is called **regular** if the system (1.2) has a unique continuously differentiable solution that satisfies $\mathbf{x}(t_0) = \mathbf{x}_0$. A point \mathbf{x}_0 is said to be **singular** if it is not regular.

considering in this module (but, in general, this is by no means a trivial task). The idea is to consider a family of *level curves* defined by a positive-definite function of the form

$$V(x_1, x_2) = Ax_1^2 + Bx_2^2,$$

for suitably chosen $A, B \in \mathbb{R}$. Such curves correspond to the set of points

$$\mathcal{L}_\alpha := \{(x_1, x_2) \in \mathbb{R}^2 : Ax_1^2 + Bx_2^2 = \alpha\}.$$

If $A = B$ these level sets represent a family of concentric circles centred at the origin, while for $A \neq B$ they define a family of concentric ellipses, again, centred at the origin. We then look for $0 < \alpha < \beta$ such that

$$\dot{V}(x_1, x_2) \begin{cases} > 0, & \text{on } \mathcal{L}_\alpha, \\ < 0, & \text{on } \mathcal{L}_\beta. \end{cases}$$

This will ensure that solutions starting in the region

$$\mathcal{R} := \{(x_1, x_2) \in \mathbb{R}^2 : \alpha < Ax_1^2 + Bx_2^2 < \beta\}$$

will remain trapped in \mathcal{R} as $t \rightarrow \infty$. The next examples illustrate how this is done in practice.

Example 1: Show that the system

$$\dot{x}_1 = x_2 - x_1^3 + x_1, \quad \dot{x}_2 = -x_1 - x_2^3 + x_2 \tag{1.8}$$

has at least one closed path in the phase plane.

SOLUTION: Note that this autonomous dynamical system is of the form (1.2), with

$$F_1(x_1, x_2) \equiv x_2 - x_1^3 + x_1 \quad \text{and} \quad F_2(x_1, x_2) \equiv -x_1 - x_2^3 + x_2.$$

Let us start by considering the positive-definite function

$$V(x_1, x_2) = x_1^2 + x_2^2.$$

Then,

$$\begin{aligned} \dot{V}(x_1, x_2) &= (\nabla V) \cdot \mathbf{F} = F_1 \frac{\partial V}{\partial x_1} + F_2 \frac{\partial V}{\partial x_2} \\ &= 2x_1(x_2 - x_1^3 + x_1) + 2x_2(-x_1 - x_2^3 + x_2) \\ &= 2[x_1^2 + x_2^2 - (x_1^4 + x_2^4)]. \end{aligned} \tag{1.9}$$

We aim to determine a circle \mathcal{C}_α , such that $\dot{V}(x_1, x_2) > 0$ for $(x_1, x_2) \in \mathcal{C}_\alpha$. To this end, recall that $x_1^2 + x_2^2 = \alpha$ and take into account that

$$x_1^4 + x_2^4 = (x_1^2 + x_2^2)^2 - 2x_1^2x_2^2 = \alpha^2 - 2x_1^2x_2^2.$$

Therefore, with the help of (1.9), we can write

$$\dot{V}(x_1, x_2) = 2(\alpha - \alpha^2 + 2x_1^2x_2^2) \geq 2\alpha(1 - \alpha) > 0,$$

provided that we choose $\alpha \in \mathbb{R}$ so that $0 < \alpha < 1$; take $\alpha = 9/10$ (for example). The trajectories of (1.8) starting inside the disk $x_1^2 + x_2^2 < \alpha$ will cross the circle \mathcal{C}_α by moving away from the origin.

Next, we are going to look for a second circle, \mathcal{C}_β (say), such that trajectories starting outside the disk $x_1^2 + x_2^2 > \beta$ will cross the circle in the opposite direction (i.e., moving toward the origin as t increases). To find such a circle, let us note that

$$2(x_1^4 + x_2^4) \geq (x_1^2 + x_2^2)^2 \quad \implies \quad x_1^4 + x_2^4 \geq \frac{1}{2}(x_1^2 + x_2^2)^2.$$

Using again (1.10) we have

$$\dot{V}(x_1, x_2) \leq 2 \left[x_1^2 + x_2^2 - \frac{1}{2}(x_1^2 + x_2^2)^2 \right] = \beta(2 - \beta), \quad \text{on } \mathcal{C}_\beta.$$

If we choose $\beta > 2$ then $\dot{V} < 0$ on \mathcal{C}_β ; take $\beta = 21/10$ (for instance).

The annular region between the two circles found above,

$$\mathcal{R} \equiv \left\{ (x_1, x_2) \in \mathbb{R}^2 : \frac{9}{10} < x_1^2 + x_2^2 < \frac{21}{10} \right\},$$

is a trapping region for the given dynamical system. Since it contains no equilibrium points, the Poincaré-Bendixson Theorem assures us that there must be at least one periodic trajectory in \mathcal{R} .

Example 2: Show that the equation

$$\ddot{y} + (2y^2 + 3\dot{y}^2 - 1)\dot{y} + y = 0, \quad (1.10)$$

has at least one periodic solution.

SOLUTION: To apply the Poincaré-Bendixson Theorem here, the second-order ODE (1.10) must be written as a system of two first-order differential equations. As usual, let us introduce $x_1 := y$ and $x_2 := \dot{y}$, so that

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -(2x_1^2 + 3x_2^2 - 1)x_2 - x_1.$$

Note that this is of the form $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, with

$$F_1(x_1, x_2) \equiv x_2 \quad \text{and} \quad F_2(x_1, x_2) \equiv -(2x_1^2 + 3x_2^2 - 1)x_2 - x_1.$$

As in the previous example, consider the positive-definite function $V(x_1, x_2) = x_1^2 + x_2^2$. Calculating the total time derivative of this function by making use of the above system, we find

$$\dot{V}(x_1, x_2) = (\nabla V) \cdot \mathbf{F} = F_1 \frac{\partial V}{\partial x_1} + F_2 \frac{\partial V}{\partial x_2}$$

$$\begin{aligned}
&= 2x_1x_2 + 2x_2[-(2x_1^2 + 3x_2^2 - 1)x_2 - x_1] \\
&= 2x_2^2(1 - 2x_1^2 - 3x_2^2). \tag{1.11}
\end{aligned}$$

First, let us identify a suitable circle \mathcal{C}_α such that $\dot{V}(x_1, x_2) > 0$ for all $(x_1, x_2) \in \mathcal{C}_\alpha$. Since $x_1^2 + x_2^2 = \alpha$,

$$\dot{V}(x_1, x_2) = 2x_2^2[1 - 3(x_1^2 + x_2^2) + x_1^2] \geq 2x_2^2(1 - 3\alpha) > 0,$$

provided that we choose $\alpha \in \mathbb{R}$ such that $\alpha < 1/3$; take $\alpha = \frac{1}{3} - \frac{1}{10} = \frac{7}{30}$ (for instance). The trajectories of (1.10) starting inside the disk $x_1^2 + x_2^2 < 7/30$ will cross the circle \mathcal{C}_α by moving away from the origin.

The second task is to find another circle, \mathcal{C}_β (say), such that trajectories starting outside the disk $x_1^2 + x_2^2 > \beta$ will cross the circle \mathcal{C}_β in the opposite direction. To this end, by using again (1.11), we find

$$\dot{V}(x_1, x_2) = 2x_2^2[1 - 2(x_1^2 + x_2^2) - x_2^2] \leq 2x_2^2(1 - 2\beta) < 0, \quad \text{on } \mathcal{C}_\beta,$$

provided that $\beta > 1/2$; take $\beta = \frac{1}{2} + \frac{1}{10} = \frac{3}{5}$ (for instance).

In conclusion, the region between the two circles found above,

$$\mathcal{R} \equiv \left\{ (x_1, x_2) \in \mathbb{R}^2 : \frac{7}{30} < x_1^2 + x_2^2 < \frac{3}{5} \right\},$$

is a trapping region for (1.10). Since it contains no equilibrium points, the Poincaré-Bendixson Theorem assures us that there must be at least one periodic trajectory in this region. This completes the solution of this example.

The next example requires considerably more ingenuity.

Example 3: Show that the system

$$\begin{cases} \dot{x}_1 = x_1 - x_2 - (x_1^2 + \frac{3}{2}x_2^2)x_1 \\ \dot{x}_2 = x_1 + x_2 - (x_1^2 + \frac{1}{2}x_2^2)x_2 \end{cases} \tag{1.12}$$

has at least one periodic solution.

SOLUTION: The main strategy is exactly the same as in the previous two examples. We start by considering the function $V(x_1, x_2) = (x_1^2 + x_2^2)/2^2$. It is easy to check that

$$\dot{V}(x_1, x_2) = x_1^2 + x_2^2 - x_1^4 - \frac{1}{2}x_2^4 - \frac{5}{2}x_1^2x_2^2. \tag{1.13}$$

Because of lack of symmetry in this result, we re-write the right-hand side of (1.13) in terms of the polar coordinates

$$x_1 = r \cos \theta, \quad x_2 = r \sin \theta.$$

Using the basic trig formulae

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta) \quad \text{and} \quad \cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta),$$

²The $\frac{1}{2}$ factor is introduced here for convenience

we find

$$\dot{V}(x_1, x_2) = r^2 - r^4 \left[1 + \frac{1}{4} \cos 2\theta - \frac{1}{4} \cos^2 2\theta \right]. \quad (1.14)$$

Next, we want to identify a value of $r > 0$ for which the right-hand side of (1.14) is strictly positive. This is equivalent to finding a positive r for which

$$\cos^2 2\theta - \cos 2\theta + 4 \left(\frac{1}{r^2} - 1 \right) > 0, \quad \text{for all } 0 \leq \theta < 2\pi.$$

We can think of the expression on the left-hand side of this inequality as a quadratic in $\cos 2\theta$. Since the coefficient of the quadratic term is positive, if the corresponding discriminant (Δ , say) is negative then the inequality will be satisfied. Thus,

$$\Delta \equiv 1 - 16 \left(\frac{1}{r^2} - 1 \right) < 0 \quad \implies \quad r < \frac{4}{\sqrt{17}}.$$

The work done above shows that if $0 < \alpha < \frac{16}{17}$, then the trajectories of (1.12) will cross the circle \mathcal{C}_α moving away from the origin.

It remains to show that it is possible to find a suitable $r > 0$ for which the right-hand side of (1.14) is strictly negative. By following the same reasoning as above we are looking for at least one $r > 0$ such that

$$\cos^2 2\theta - \cos 2\theta + 4 \left(\frac{1}{r^2} - 1 \right) < 0, \quad \text{for all } 0 \leq \theta < 2\pi. \quad (1.15)$$

Since $-1 \leq \cos 2\theta \leq 1$, we can write $\cos^2 2\theta - \cos 2\theta \leq 2$, and we are asking if there are any values of r for which

$$2 + 4 \left(\frac{1}{r^2} - 1 \right) < 0 \quad \implies \quad r > \sqrt{2}.$$

This shows that for $r > \sqrt{2}$ the inequality (1.15) is satisfied. It also means that for $\beta > 2$ the trajectories of (1.12) will cross the circle \mathcal{C}_β moving toward the origin.

In conclusion, the annular region between the aforementioned circles \mathcal{C}_α and \mathcal{C}_β can be used to play the role of \mathcal{R} in the Poincaré-Bendixson theorem, etc.

Theorem 3 (Bendixson): Let $\mathcal{D} \subset \mathbb{R}^2$ be a simply connected³ region in which the vector field $\mathbf{F}(\mathbf{x}) = (F_1(x_1, x_2), F_2(x_1, x_2))$ has the property that

$$\nabla \cdot \mathbf{F} \equiv \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}$$

is of constant sign. Then the system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ has *no* closed trajectories *wholly* contained in \mathcal{D} .

This theorem can be justified by using a standard result in vector calculus:

³It will be sufficient for our purposes to recognise that such a region of the plane is one that has no 'holes' in it.

Green's Theorem (in the plane): Let the real-valued functions $P(x_1, x_2)$ and $Q(x_1, x_2)$ have continuous first partial derivatives in a simply connected region $\mathcal{D} \subset \mathbb{R}^2$ bounded by a simple curve \mathcal{C} . Then

$$\oint_{\mathcal{C}} P dx_1 + Q dx_2 = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2} \right) dx_1 dx_2, \quad (1.16)$$

where the integration along \mathcal{C} is done in an anti-clockwise direction.

To prove Bendixson's Theorem we assume that a limit cycle \mathcal{C} of period $T > 0$ exists for the given system. Let $P := -F_2$ and $Q := F_1$ in (1.16), so

$$\oint_{\mathcal{C}} F_1 dx_2 - F_2 dx_1 = \iint_{\mathcal{D}} \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} \right) dx_1 dx_2 \neq 0. \quad (1.17)$$

We can also re-write the left-hand side of this last equation by taking into account that $\dot{x}_j = F_j$ for $j = 1, 2$:

$$\begin{aligned} \oint_{\mathcal{C}} F_1 dx_2 - F_2 dx_1 &= \int_0^T (F_1 \dot{x}_2 - F_2 \dot{x}_1) dt \\ &= \int_0^T (\dot{x}_1 \dot{x}_2 - \dot{x}_2 \dot{x}_1) dt = 0, \end{aligned} \quad (1.18)$$

where we have used $dx_j = \dot{x}_j dt$ ($j = 1, 2$). The results in (1.17) and (1.18) are in contradiction with each other. Hence, the closed trajectory \mathcal{C} cannot exist and this completes the proof of Theorem 3.